„Valuation of Mortgage-Backed Securities“

A Simulation-based Approach.

Florian C. Leisch

Based on the diploma thesis supervised by Robert Tichy and Hansjörg Albrecher at the Graz University of Technology

Mortgage-Backed Securities(MBS) are fixed income securities which are collateralized by a pool of (residential) mortgage loans. The valuation of these instruments is made difficult by the varying amounts of monthly cash-flows which are due to prepayments. In this paper, an interest rate path Monte-Carlo algorithm is presented which projects prepayments conditional on a simulated interest rate path. Special attention is paid to modeling correlated defaults with a factor-copula model. Refinancings are projected using a threshold model which relies on Generalized-Inverse Gaussian distributions which evolve over the life of the pool. Housing turnover is quantified by constructing stochastic processes conditional on the interest rate, which are calibrated to historical data.

Keywords: Mortagage-Backed Securitiy, Copula, Generalized Inverse Gaussian Distribution, Interest Rate Paths, Monte-Carlo

Florian Leisch worked as Risk Analyst at Raiffeisen Capital Management before he changed to the central bank of the Republic of Austria (OeNB). This article is based on his diploma thesis which was completed and presented at TU Graz in June 2006. The views and opinions formulated in this article are the author's own and are not expressed on behalf of Oesterreichische Nationalbank (OeNB).

1 Introduction

The steady rise of Mortgage-Backed Securities(MBS) and their steep fall in 2007 as well as the ensuing market turmoil have made this asset class a center of attention for the future development of global financial markets. The events on the structured credit market in 2007 and 2008 have had a significant impact on banks, insurances and other financial institutions. Thus the issues of valuation, risk management and stress-testing will deserve some attention, as far as the future development of MBS and other securitized debt products is concerned.

Basically, MBS are securities entitled to the cash flows from a certain pool of (residential) mortgage loans. Hence they are a special case of Asset-Backed Securities which are securities backed by a pool of very general assets which might be as diverse as student loans, auto loans or credit card receivables. MBS are structured as follows: The cash flows from the underlying pool of mortgage loans, i.e. interest and principal payments, are passed through to investors after subtracting a certain servicing fee. This is the simplest way payments can be channeled to investors, the so called **pass-through MBS**. Another possibility is to channel the payments to investors according to specified rules, thus creating more complicated, structured products as e.g. Collateralized Mortgage Obligations(CMO). The focus will be put on the basic case of pass-through MBS, for further information on CMOs and related products see [6].

1.1 Development of the Market

The development of the MBS market was actively supported by the US government by means of the creation of three housing agencies: *Ginnie Mae(Government National Mortgage Association, GNMA), Fannie Mae(Federal National Mortgage Association, FNMA)and Freddie Mac(Federal Home Loan Mortgage Corporation, FHLMC).* The agencies buy mortgages from lenders, thus providing them with fresh capital which can be used to make new loans. The mortgage portfolios can then be securitized via issuance of an MBS constituting a liquid asset which can be traded actively. These so-called agency MBS form a substantial part of the residential MBS market. The key advantage when investing in agency MBS is - from the investor's point of view - that the agencies guarantee the timely payment of interest and principal. This means that in case of defaulting mortgagors the agency will liquidate the mortgage from the portfolio and pay the outstanding value to investors who will receive a higher than scheduled coupon. This is a major advantage over so-called private sector MBS where default risk is predominantly unabated. Despite these characteristics, agency MBS - together with the agencies - have suffered substantially in the course of the credit crisis in 2007 and 2008. In the subsequent considerations we will focus on standard pass-through agency MBS which are backed by a pool of non-assumable, fixed-rate 30-year mortgage loans.

1.2 Prepayments and the Fair Price

Let us now define what we understand to be the *fair price* of an MBS.

Definition 1.1: ***The fair price of an MBS****: Let there be N months to maturity and denote by $\widehat{CF}_i$ the* ***actual*** *cash flow in month $i = 1, \ldots, N$. Then the fair price of the MBS is defined as:*

$$P_F = \sum_{i=1}^{N} v(i,s)\widehat{CF}_i \tag{1}$$

where $v(i,s)$ are the discount factors for some given spread s and some given interest path which we assume to be the actual one as well. I.e. if we let $f_1, \ldots, f_n$ be the one-month forward rates in month $i = 1, \ldots, n$, then the discount rate for month n for a given s is:

$$v(n,s) = \frac{1}{(1+f_1+s)\cdots(1+f_n+s)}. \tag{2}$$

Obviously enough, the fair price of an MBS at any time in its life is given by the present value of its cash flows. And that is exactly where the fundamental problem of MBS valuation arises: *one does not know in advance the quantity of the cash flows over the remaining life of the security.*
Let us therefore consider a certain interest rate path and let us assume that prepayment speeds can be projected conditional on the prevailing interest rates. The fair price of an MBS can therefore be approximated by computing the prices over a sufficiently large number of interest rate paths and taking the average over the calculated prices. This will also be the fundamental idea of the valuation algorithm outlined in this paper. Three of the most important reasons for prepayments are home sales, refinancings and defaults (see [6]). These economic factors will now be entered into a mathematical model which will finally allow us to compute prices for MBS. For more detailed results on the prepayment behavior of mortgagors we refer to literature (see [6], [8] and [14]).

2 The Mortgage Rate Model

Recent results in interest rate modeling show, that very reliable models may be achieved through use of Lévy processes (see e.g. [4], [9], [12] or [13]). We shall follow the Heath-Jarrow-Morton approach for constructing spot rates and forward rates as it is outlined in Özkan [12] and the resulting interest rates will then be used to construct the MBS discounting factors and the mortgage rate process. A detailed introduction to Lévy processes and their properties can be found in e.g. in Applebaum [2].

The default free term structure will be specified following the Heath-Jarrow-Morton (HJM) model as it is also used in [4] and [12]. The following Assumptions will be made. Let T^* be a fixed time horizon and let us assume that for every $T \in [0, T^*]$ there exists a risk free zero-coupon bond with maturity T. The dynamics of the model shall depend on a Lévy process L specified by the Lévy triple (b, A, ν) (see [2]) on a probability space $(\Omega, \mathfrak{A}, P)$ with a filtration $\mathfrak{F} = (\mathfrak{F}_t)$ which is defined as $\mathfrak{F} = \sigma(L)$, the filtration which is generated by the Lévy process. We postulate that the dynamics of the instantaneous forward rate can be modeled as follows (see [9]):

Assumption 2.1: ***Instantaneous Forward Rate:***
For $T \in [0, T^*]$ the forward rate $f(t,T)$ is modeled as:

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds - \int_0^t \sigma(s,T)dL_s \tag{3}$$

$(0 \le t \le T)$, α, σ are stochastic processes on $\mathbb{R}$ which fulfill the following conditions:

1. $(\omega, s, T) \mapsto \alpha(\omega, s, T)$ *and* (ω, s, T) *are measurable w.r.t.* $\mathfrak{P} \otimes \mathcal{B}[0, T^*]$

2. $s > T: \ \alpha(\omega, s, T) = 0$ *and* $\sigma(\omega, s, T) = 0$

3. $\sup_{s, T \leq T^*}(|\alpha(\omega, s, T)| + |\sigma(\omega, s, T)|) < \infty$

These conditions ensure that a joint version of all $f(t, T)$ *exists, such that* $(\omega, t, T) \mapsto f(t, T)(\omega)\mathbf{1}_{\{t \leq T\}}$ *is* $\mathfrak{O} \otimes \mathcal{B}[0, T^*]$ *measurable.* $\mathfrak{P}$ *and* $\mathfrak{O}$ *denote the predictable and the optional* σ*-algebra on* $\Omega \times [0, T^*]$ *and* $\mathcal{B}[0, T^*]$ *is the Borel* σ*-algebra (see [9]).*

The concept of forward rates is directly linked to zero-coupon bond prices. Consequently, the price of a zero coupon bond at time t with maturity T, $B(t, T)$ may be recovered from forward rates by the following formula:

$$B(t, T) := \exp\left(-\int_t^T f(t, u)du\right) \quad \forall t \in [0, T]. \tag{4}$$

The *short rate* $r(t)$ at time t is the interest rate to be paid or received over an infinitesimal interval, it can obviously be obtained as:

$$r(t) := f(t, t). \tag{5}$$

2.1 The MBS Discounting Factor

From the investor's point of view, an MBS is seen as yet another bond albeit with special properties as e.g. varying cahs flows. This is the reason why we recall the general Definition of the risk free bond price given in (4). An MBS is in general not a risk free security and we have to take that into account, but the discounting factor follows the same principles as in the risk free case.

Definition 2.1: *Let the forward rates* $f(t, T)$ *be defined as given in* (3) *and let us assume* $s_1(t)$ *to be a deterministic, bounded and continuous spread function. Then the MBS discounting factor at time* t *for payments at time* T *for a given spread function is defined by:*

$$v(t, T, s_1) := \exp\left(-\int_t^T (f(t, u) + s_1(u))du\right). \tag{6}$$

Let $t \leq T_1 < \ldots < T_n = T^*$ be the remaining future cash flow times of an MBS pool and let CF_i be the corresponding cash flows, $i = 1, \ldots, n$. Applying Definition 2.1 to the characterization of the fair price given in Definition 1.1 we obtain:

$$V_{MBS}(t, s_1) = \sum_{i=1}^{n} v(t, T_i, s_1)CF_i. \tag{7}$$

Hence the MBS are priced at a spread over the risk free yield curve which is the kind of behavior that can also be observed in the markets (see [6]).

2.2 The Mortgage Rate

The approach which is going to be used for the construction of the mortgage rate is slightly different. Let us start with a few elementary observations. The short rate r describes the interest which has to be paid for lending or investing money from now on over an infinitesimal time period. Investing the amount V_0 in the risk free savings account for a time period $[0, t]$ will yield the value process V_t:

$$V_t = V_0 \exp\left(\int_0^t r(s)ds\right) = V_0 \exp\left(\int_0^t f(s,s)ds\right) \tag{8}$$

$$= V_0 \exp\left(\int_0^t \left[f(0,s) + \int_0^u \alpha(v,u)dv - \int_0^u \sigma(v,u)dL_v\right] ds\right). \tag{9}$$

Returning to the mortgage rate model we shall specify a time horizon $[0, 2T^*]$ which is twice the life of the mortgages in the pool given by T^*. At time 0 a potential home-buyer can obtain a non-assumable residential mortgage with life T^* which is prepayable at any time at rate $\tilde{m}$. The aim is to specify the process $m = (m_t)_{0 \le t \le T^*}$ where m_t is the mortgage rate offered at time t for a mortgage which meets the same conditions as the original one. At this point we may construct the following model for m.

Let $m_0 = \tilde{m}$ be the starting point of the mortgage rate process, let us again assume s_2 to be a deterministic, bounded and continuous spread function which reflects the premium a mortgagor has to pay over the risk free rate. Starting at time 0 the following notion holds:

$$\exp(\tilde{m}T^*) = \exp\left(\int_0^{T^*} (r(u) + s_2(u))du\right). \tag{10}$$

This means that the value of capital invested at time 0 up to time T^* at rate $\tilde{m}$ must equal the capital invested in the risk free savings account plus a spread. Taking the logarithm of Equation 10 yields:

$$\tilde{m}T^* - \int_0^{T^*} r(u)du = \int_0^{T^*} s_2(u)du. \tag{11}$$

This notion can be used to specify the spread function. The same notion holds at time t.

$$\exp(m_t T^*) = \exp\left(\int_t^{t+T^*} (r(u) + s_2(u))du\right). \tag{12}$$

Hence one obtains:

Definition 2.2: ***The Mortgage Rate Process:***
The mortgage rate process $m = (m_t)_{0 \le t \le T^*}$ is defined as:

$$m_t = \frac{1}{T^*} \int_t^{t+T^*} (r(u) + s_2(u))du. \tag{13}$$

A comprehensive interest rate model has now been established which provides the elementary tools for setting up a prepayment model.

3 Turnover-Related Prepayments

The following economic factors can be identified as influential on the decision of purchasing a new home: the average income, the prevailing mortgage rate and the price of an average American home (see [6]). These factors and their dynamics will now be incorporated in the model in order to compute an affordability measure which can finally be translated into a prepayment speed.

Let $m = (m_t)_{0 \leq t \leq T^*}$ be the mortgage rate process during the life of the MBS pool, i.e until time T^*. Let furthermore $I = (I_t)_{0 \leq t \leq T^*}$ be the income process and $H = (H_t)_{0 \leq t \leq T^*}$ the home price process where I and H are initialized by $I_0 = \tilde{I}_P$ and $H_0 = \tilde{H}_P$, $\tilde{I}_P$ being the average net income of all obligors in our pool and $\tilde{H}_P$ the average house price in the pool. One postulates that both income and house prices are subject to change over time and that the resulting dynamics are predominantly due to the mortgage rates. Furthermore it is claimed that the mortgage rates do mirror the economic situation in the following way. Mortgage rates rise and fall in line with the general interest rate level and we assume the following: Rising interest rates reflect a strong economic situation with healthy consumer demand whereas low interest rates are linked to a weak economic situation and low interest rates are intended to stimulate demand. Hence one can reasonably claim the processes I and H to follow the dynamics

$$dI_t = \kappa_1 I_{t-} dm_t + \sigma_1 dB_t. \tag{14}$$

For $\kappa_{1,2}, \sigma_{1,2} \in \mathbb{R}$ and $B = (B_t)_{0 \leq t \leq T^*}$ a standard Brownian Motion. The dynamics of the house prices are given analogously by:

$$dH_t = \kappa_2 H_{t-} dm_t + \sigma_2 dB_t. \tag{15}$$

For some given amount K which shall be taken out as a loan and which shall be paid back according to the constant process $\tilde{M}_t = \tilde{M} \ \forall t \in [0, T^*]$, and for a continuous interest rate r, $\tilde{M}$ is given by

$$K = \tilde{M} \int_0^{T^*} e^{-ru} du. \tag{16}$$

Hence one can accordingly compute a process $M = (M_t)_{0 \leq t \leq T^*}$, the monthly mortgage payment process as follows. If we fix t, then M_t is given by:

$$M_t = \frac{\alpha H_t}{\int_0^{T^*} e^{-m_t u} du}, \tag{17}$$

where $\alpha \in [0, 1]$ is the factor that represents the percentage of the new house which is financed by the loan. Thus we define the affordability measure as follows:

Definition 3.1: *The affordability measure A_t at time t is defined as*

$$A_t = \frac{I_t}{M_t}, \tag{18}$$

i.e. it is the ratio of the average income at time t and the monthly mortgage payments due on a mortgage taken out at time t. We call the process $A = (A_t)_{0 \leq t \leq T^*}$ affordability process.

What remains to be done is the Definition of the desirability measure.

Definition 3.2: *We define the desirability measure J_t at time t as a linear combination of the processes $(m_t)_{0\leq t\leq T^*}$, $(I_t)_{0\leq t\leq T^*}$ and $(H_t)_{0\leq t\leq T^*}$.*

$$J_t = \alpha_1 m_t + \alpha_2 I_t + \alpha_3 H_t. \tag{19}$$

The corresponding process is denoted by $J = (J_t)_{0\leq t\leq T^*}$. The real-valued parameters α_i can be calibrated to available data.

This setup allows us to describe the behavior of potential home buyers which can be observed in the markets.

4 A Threshold Model for Refinancings

The refinancing model will be based on the idea of calculating a prepayment measure at every time t which is conditional on the prevailing mortgage rate. A similar approach is outlined in [6]. We will assume that every mortgagor possesses a distinctive refinancing propensity which is modeled via a threshold value ζ_i for all $i = 1, \ldots, n$ mortgagors in the pool. This means that the mortgagor is very likely to refinance, once the refinancing incentive measure is greater than or equal to the threshold value.

4.1 The Refinancing Incentive

As in the cases treated before, it will be assumed that the life of the MBS pool is given by the time interval $[0, T^*]$ and that the mortgage rate process is given by $m = (m_t)_{0\leq t\leq T^*}$. For a certain loan with constant payments of $\tilde{M}$ at a constant mortgage rate of $\tilde{m}$ one can give the outstanding loan balance at time t, $b = (b_t)_{0\leq t\leq T^*}$, as the present value of the remaining payments:

$$b_t = \tilde{M} \int_t^{T^*} e^{-\tilde{m}s} ds. \tag{20}$$

In case of a refinancing at time t the mortgagor would take out a new loan covering the outstanding balance b_t at the prevailing mortgage rate m_t. Assuming that this loan will be paid back with constant payments too, the payments for the new loan can be given as:

$$\mu_t = \frac{b_t}{\int_t^{T^*} e^{-m_t s} ds}. \tag{21}$$

That means that the process $\mu = (\mu_t)_{0\leq t\leq T^*}$ describes the constant mortgage payments that have to be made if the remaining loan balance at time t is refinanced given the prevailing market conditions and assuming that the maturity of the loan stays the same. As a consequence we can compute an incentive process $\iota = (\iota_t)_{0\leq t\leq T^*}$ which is defined as follows:

$$\iota_t = \left(\tilde{M} \int_t^{T^*} ds - \mu_t \int_t^{T^*} ds - \gamma b_t - \phi \right)_+, \tag{22}$$

i.e. the sum of mortgage payments in case of a refinancing and the variable refinancing costs γb_t as well as the fixed refinancing costs ϕ are subtracted from the sum of payments due on the existing mortgage. If this expression is positive it means that refinancing the existing loan is attractive, if it is negative, the incentive to refinance is set to zero. Furthermore we define the process $\rho = (\rho_t)_{0 \leq t \leq T^*}$ which gives the ratio between $\tilde{M}$ and μ_t:

$$\rho_t = \frac{\tilde{M}}{\mu_t}. \tag{23}$$

If the ratio is greater than one, it is attractive to refinance, yet the process ρ does not take refinancing costs into account. The absolute refinancing incentive is now defined as follows.

Definition 4.1: *Let the processes $\iota = (\iota_t)_{0 \leq t \leq T^*}$ and $\rho = (\rho_t)_{0 \leq t \leq T^*}$ be defined as specified in Equations* (22) *and* (23)*. Then the absolute refinancing incentive is defined by $\eta = (\eta_t)_{0 \leq t \leq T^*}$:*

$$\eta_t := \begin{cases} \kappa \log(\iota_t)\rho_t, & \iota_t > 0; \\ \epsilon, & \iota_t = 0. \end{cases} \tag{24}$$

In Definition 24, $\kappa > 0$ is a scaling parameter and $1 > \epsilon > 0$ is a sufficiently small value that represents the case that some people refinance even if it is not reasonable to do so (see [15]).

4.2 The Threshold Distribution

Let there be $i = 1, \ldots, n$ mortgagors in the pool, each of them possesses a threshold value ζ_i which describes the debtor's refinancing propensity. We assume that these threshold values follow a Generalized Inverse Gaussian distribution (see e.g. Bibby and Sørensen [3]) whose parameters change due to the changes in the mortgage rate m and due to the age of the pool. Hence for every time t it holds that $\zeta_i(t) \sim GIG(\lambda(t), \delta(t), \gamma(t))$. The parameters $\lambda(t)$, $\delta(t)$ and $\gamma(t)$ are defined implicitly as follows. The initial distribution $GIG(\lambda(0), \delta(0), \gamma(0))$ is fitted to historical observations and thus we obtain the initial mean and variance of the distribution as well as the parameter $\omega(0)$, where $\omega(t)$ is defined by:

$$\omega(t) = \delta(t)\gamma(t). \tag{25}$$

Let us denote the mean and variance of the threshold distribution at time t by $E(t)$ and $V(t)$ respectively. The initial values $E(0)$ and $V(0)$ are known. Let us furthermore assume that the variance can be held constant throughout the life of the pool. Since it is our goal to model observable refinancing-characteristics, we claim that the mean of the threshold distribution evolves according to the following law:

$$E(t) = E(t-) + \kappa_1 \Delta m_t + \kappa_2 \Delta t, \;\; \kappa_i \in \mathbb{R}. \tag{26}$$

This means that the mean increases proportionally to the length of the time interval and conditional on the increase or decrease of the mortgage rate in this time interval. Similarly we also demand that

$\omega(t)$ grows as the pool matures, essentially following the same dynamics as $E(t)$ does:

$$\omega(t) = \omega(t-) + \kappa_3 \Delta m_t + \kappa_4 \Delta t, \;\; \kappa_j \in \mathbb{R}. \tag{27}$$

Consequently, the parameters $\lambda(t)$, $\delta(t)$ and $\gamma(t)$ are defined implicitly as the triplet solving the following system (where $K_\lambda(x)$ is the modified Bessel function of the third kind with index $\lambda \in \mathbb{R}$):

$$(\delta(t)/\gamma(t)) \frac{K_{\lambda(t)+1}(\omega(t))}{K_{\lambda(t)}(\omega(t))} = E(t) \tag{28}$$

$$(\delta(t)/\gamma(t))^2 \left(\frac{K_{\lambda(t)+2}(\omega(t))}{K_{\lambda(t)}(\omega(t))} - \frac{K^2_{\lambda(t)+1}(\omega(t))}{K^2_{\lambda(t)}(\omega(t))} \right) = V(t) \tag{29}$$

$$\delta(t)\gamma(t) = \omega(t). \tag{30}$$

This system yields as a solution triplets of parameters which fully specify the threshold distribution at time t. Modeling the evolution of the threshold distribution as specified above will ensure that refinancing behavior changes in the same way as can be observed in the markets. At every time t the absolute refinancing incentive is given by η_t and the probability that the threshold value of the i-th mortgagor lies below η_t, i.e. the probability that the obligor refinances is given by $p_t = P(\zeta_i(t) \leq \eta_t) = F_t(\eta_t)$ where F_t is the distribution function of the GIG distribution as observed at time t. Let $B_t^{(n)}$ be the total balance of the pool at time t with n mortgagors still in the pool, then we obtain the projected cash flows due to refinancings as $CF_{ref}(t) = p_t B_t^{(n)}$. The aspect that the mean increases more quickly if mortgage rates drop mirrors the kind of refinancing behavior that can be observed in the market.

5 Default-Related Prepayments

We will use a factor copula approach for modeling correlated defaults (see e.g. [10], [16]). A comprehensive introduction to copulae and their applications can be found in Nelsen [11]. Let there be n different mortgagors in the pool, each of them has an intrinsic default-time distribution $\tau_i \sim F_i$, where τ_i denotes the default time of the i-th obligor. Let for any obligor i denote the default indicator at time t by $X_i^{(t)}$. We assume that the default indicator is given by the following factor model:

$$X_i^{(t)} = \beta_i V + V_i \sqrt{1 - \beta_i^2}, \tag{31}$$

where $V, V_i \sim N(0,1)$ and independent and where β_i is the factor linking V and V_i. V is the general indicator which is the same for all debtors in the pool and V_i is the individual indicator. Such a factor model was introduced in Laurent and Gregory [10] and further developed in Will [16]. Here their model is enhanced by the aspect of time dependency. For any obligor i we can introduce a random variable $\tau_i \sim F_i^{(t)}$ which describes the default time and the probability that the i-th debtor defaults until time t is $P(\tau_i \leq t) = F_i^{(t)}(t) =: u_i^{(t)}$. At this point the time-dependency concept comes into play. Recalling the fact that the mortgage rates mirror the economic situation, it is assumed that the default time distribution is conditional on the prevailing mortgage rate at time t. The influence

of the mortgage rate on the default distribution will therefore be modeled as follows. Assuming that a healthy economic environment which is reflected by a high mortgage rate prevails, a certain obligor i in the mortgage pool is very likely to enjoy a better financial situation than he does on average. Claiming that the distribution $F_i^{(t)}$ is a right-skewed distribution on the positive half-axis, this temporary improvement in credit quality will be captured by adjusting the parameters such that probability mass is shifted from the peak to the tail in a good economic environment and the other way around in a bad economic situation. Thus capturing the interest rate path inherent characteristics ensures that all parts of the prepayment model are related in the desired way.

Returning to the formulation of the default model we can henceforth derive that the i-th obligor defaults if:

$$X_i^{(t)} = \beta_i V + V_i \sqrt{1-\beta_i^2} \leq \Phi^{-1}(u_i^{(t)}), \tag{32}$$

or, if we put Equation (32) in a slightly different manner (see [16]):

$$V_i \leq \frac{\Phi^{-1}(u_i^{(t)}) - \beta_i V}{\sqrt{1-\beta_i^2}} \Leftrightarrow \tau_i \leq t. \tag{33}$$

Conditional on some realization of V the default probability $p_t^{i|V}$ can be given by:

$$p_t^{i|V} = \Phi\left(\frac{\Phi^{-1}(u_i^{(t)}) - \beta_i V}{\sqrt{1-\beta_i^2}}\right) = P(\tau_i \leq t|V). \tag{34}$$

The unconditional probability can thus be obtained by integration:

$$p_t^i = E(p_t^{i|V=v}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Phi\left(\frac{\Phi^{-1}(u_i^{(t)}) - \beta_i v}{\sqrt{1-\beta_i^2}}\right) e^{-\frac{v^2}{2}} dv. \tag{35}$$

We are now going to couple these $X_i^{(t)}$ by means of a copula C, such that:

$$C^{(t)}(u_1^{(t)}, \ldots, u_n^{(t)}) = P(X_1^{(t)} \leq \Phi^{-1}(u_1^{(t)}), \ldots, X_n^{(t)} \leq \Phi^{-1}(u_n^{(t)})). \tag{36}$$

This can be done by the normal copula as specified e.g. in [16]. Due to the independence between V and V_i, $i = 1, \ldots, n$, the covariance of $X_i^{(t)}, X_j^{(t)}$ is given by:

$$Cov(X_i^{(t)}, X_j^{(t)}) = \beta_i \beta_j. \tag{37}$$

Furthermore, the Gaussian factor copula is given by:

$$C^{(t)}(u_1^{(t)}, \ldots, u_n^{(t)}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\prod_{i=1}^{n} \Phi\left(\frac{\Phi^{-1}(u_i^{(t)}) - \beta_i v}{\sqrt{1-\beta_i^2}}\right)\right) e^{-\frac{v^2}{2}} dv. \tag{38}$$

If we let $t_1, \ldots, t_n$ be different default times for each obligor, then the joint distribution is clearly given by:

$$F(t_1, \ldots, t_n) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\prod_{i=1}^{n} \Phi \left(\frac{\Phi^{-1}(F_i^{t_i}(t_i)) - \beta_i v}{\sqrt{1-\beta_i^2}} \right) \right) e^{-\frac{v^2}{2}} dv. \tag{39}$$

Obviously enough we are only interested in $F(t, \ldots, t)$. By means of the Factor-Copula concept sketched above, we may estimate the probability $p_k^{(t)}$ that k debtors in the pool default before time t. Hence the expected capital lost due to default at time t is given by:

$$D_t = \sum_{k=1}^{n} k b_t p_k^{(t)}. \tag{40}$$

Calibrating this model to market data will finally provide us with a valuation framework which offers the opportunity to trace default-related speeds in a reasonable way.

Recently, considerable advances could be achieved in replacing one-factor normal models by more general ones. Albrecher, Ladoucette and Schoutens [1] propose a generic one-factor Lévy model for pricing CDOs. Previous results include Kalemanova, Schmid and Werner [7] and Guegan and Houdain [5]. In both papers Normal-Inverse-Gaussian models are used for CDO pricing.

6 Interpretation of Numeric Results

We are now going to discuss the results which were obtained by the implementation of the above model in Mathematica. The MBS pool which is considered is the Ginnie Mae 6.5% Pool Nr. 605700 which was issued on the 1st of September 2004 and which matures on the 15th of September 2034, i.e. - at the time of calculation - a remaining life of 342 months. This security was traded at \$ 103.31 per share as of April, 18th, 2006.

Simulating turnover- refinancing- and default-related cash-flows for an exemplary interest rate path, one obtains a combined prepayment scenario which is illustrated by the plots below. In Figure 1 the cash-flows due to prepayments and the scheduled payments can be seen. The area of high prepayments between months 70 and 150 is predominantly due to the strong refinancing activity at this time. The initial spike in prepayments is also partly due to refinancings and partly due to the first pickup in turnover speeds. The high prepayments up to month 60 are predominantly those caused by defaults. The small kinks in the structure of the scheduled payments result from the sudden changes in the number of debtors caused by prepayments.

We simulate $M = 200$ HJM interest rate paths and calculate the corresponding prices, the results can be seen in the table below:

	$M = 50$	$M = 100$	$M = 150$	$M = 200$	Market Price
HJM	102.871	103.232	103.353	103.348	103.31

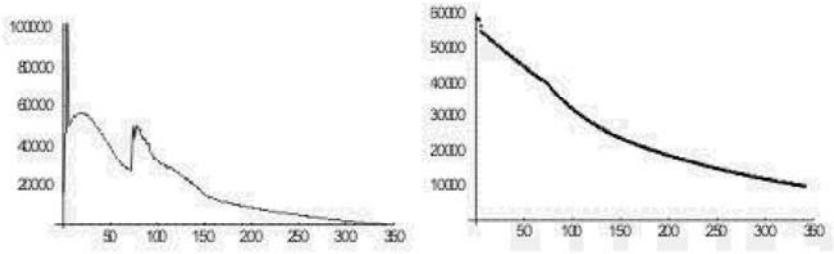

Figure 1: Prepayments and Scheduled Payments.

It can be seen that the algorithm converges reasonably well, the quality of the obtained results is furthermore underlined by the 95% confidence intervals which can be constructed for the prices given above.

	$M = 50$	$M = 100$	$M = 150$	$M = 200$
HJM	$[102.023, 103.718]$	$[102.612, 103.853]$	$[102.783, 103.942]$	$[102.876, 103.819]$

The structure of the simulated prices is illustrated by Figure 2 where the scatter-plot of the simulated prices along with the mean, confidence interval and standard deviation can be seen.

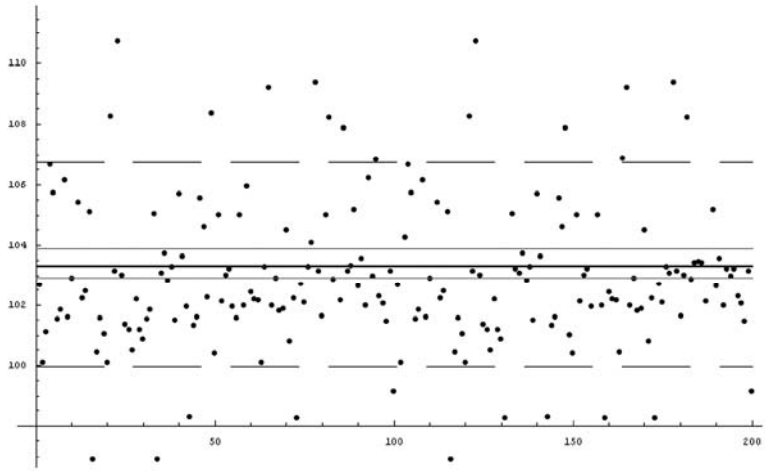

Figure 2: HJM Prices.

We can see the mean(blue/black) with its 95 % confidence interval(red) and the standard deviation(dashed). Concluding our brief discussion of numeric results, it should be stated that our algorithm yields realistic and useful results, even if only very little data is available for calibration purposes.

7 Conclusion

It can be seen in the numeric example outlined above, that the mathematical prepayment model which has been developed in this paper yields realistic results. An interest rate path Monte Carlo algorithm has been implemented which captures the influence factors housing turnover, refinancings and borrower defaults.

The resulting implementation apparently provides useful results even if the number of replications is small. The use of copula functions for default-related prepayments and the threshold model for refinancings introduce some mathematical concepts to MBS research which have - to our knowledge - not previously been used in this field.

The fundamental changes in securitized debt markets which happened between the completion of the underlying diploma thesis in 2006 and the creation of this paper in 2008 show, that research in the field of MBS valuation still deserves our attention.

References

[1] Hansjörg Albrecher, Sophie A. Ladoucette and Wim Schoutens: *A generic one-factor Lévy model for pricing synthetic CDOs*. Preprint Graz University of Technology and Katholieke Universiteit Leuven, 2006.

[2] David Applebaum: *Lévy Processes and Stochastic Calculus*. Cambridge Studies in Advanced Mathematics, Cambridge, UK, 2004.

[3] Bo Martin Bibby and Michael Sørensen: *Hyperbolic Processes in Finance*. In: Rachev, S. T. (Ed.) Handbook of Heavy Tailed Distributions in Finance. Elsevier Science, 211–248, 2003.

[4] Ernst Eberlein, Wolfgang Kluge and Philipp J. Schönbucher: *The Lévy Libor model with default risk*. Preprint, Albert-Ludwigs-Universität Freiburg im Breisgau, 2005.

[5] Dominique Guegan, Julien Houdain: *Collateralized Debt Obligations pricing and factor models: a new methodology using Normal Inverse Gaussian distributions*. Note de recherche IDHE-MORA n°07-2005, ENS Cachan, 2005.

[6] Lakhbir Hayre ed.: *The Salomon Smith Barney Guide to Mortgage Backed and Asset Backed Securities*. John Wiley & Sons, New York, 2001.

[7] Anna Kalemanova, Bernd Schmid and Ralf Werner: *The Normal Inverse Gaussian distribution for synthetic CDO pricing*. Working Paper, Risklab Germany GmbH, 2005.

[8] Andrew Kalotay, Deane Yang and Frank J. Fabozzi: *An Option-Theoretic Prepayment Model for Mortgages and Mortgage-Backed Securities*. International Journal of Theoretical and Applied Finance, Vol.7, No.8, 2004.

[9] Wolfgang Kluge: *Time-inhomogeneous Lévy processes in interest rate and credit risk models*. Dissertation, Albert-Ludwigs-Universität Freiburg im Breisgau, 2005.

[10] Jean-Paul Laurent and Jon Gregory: *Basket Default Swaps, CDO's and Factor Copulas*. Working Paper, Université de Lyon and BNP Paribas, 2002.

[11] Roger B. Nelsen: *An Introduction to Copulas*. Springer, New York, 1999.

[12] Fehmi Özkan: *Lévy Processes in Credit Risk and Market Models*. Dissertation, Albert-Ludwigs-Universität Freiburg im Breisgau, 2002.

[13] Sebastian Raible: *Lévy-Processes in Finance: Theory, Numerics and Empirical Facts*. Dissertation, Albert-Ludwigs-Universität Freiburg im Breisgau, 2000.

[14] Ehud I. Ronn, Peter D. Rubinstein and Fung-Shine Pan: *An Arbitrage-Free Estimate of Prepayment Option Prices in Fixed-Rate GNMA Mortgage-Backed Securities*. Real Estate Economics, Vol.23, No.1, 1995.

[15] Richard Stanton: *Rational Prepayment and the Valuation of Mortgage-Backed Securities*. The Review of Financial Studies, Vol.8, No.3, 1995.

[16] Bernd Will: *Valuation of Multi-Name Credit Derivatives*. MSc Thesis, Exeter College, University of Oxford, 2003.