Claim Reserving – Deterministic, Stochastic and Multivariate Methods

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Abstract

In [2] the chain–ladder method is carried out based on the assumptions that the increments and therefore the claim amounts are strict positive. The chain–ladder method considered here shows the necessary and sufficient condition when zero increments are allowed. Stochastic models underlying the deterministic method and the non-parametric model of Mack are described under this condition.

Furthermore the extension of the chain–ladder method to m subportfolios is described. The multivariate chain–ladder method consisting of n conditional linear models corresponding to the development years is described.

1 Introduction

Let $C_{j,k}$ with $j,k \in \{0,1,\ldots,n\}$ denotes the cumulative claim amount (plus estimated reserves) of accident year j and development year k. Obviously, the index

$$j \in \{0,1,\ldots,n\} \text{ refers to accident years (rows),}$$
$$k \in \{0,1,\ldots,n\} \text{ refers to development years (columns)}$$

and the $C_{j,k}$'s are observable for $j+k \leq n$ and non-observable for $j+k > n$. For $j+k=n$ we call it the actual cumulative claim amount and for $k=n$ it is called the ultimate cumulative claim amount. Besides the cumulative claim amount $C_{j,k}$ we often consider the incremental claim amount $I_{j,k}$ defined by

$$I_{j,k} := \begin{cases} C_{j,0} & \text{for } k=0, \\ C_{j,k} - C_{j,k-1} & \text{for } k \in \{1,\ldots,n\} \end{cases} \tag{1}$$

and note that

$$C_{j,k} = \sum_{l=0}^{k} I_{j,l}, \qquad j,k \in \{0,1,\ldots,n\}. \tag{2}$$

The $I_{j,k}$ can be interpreted as the amount to be paid in development year k for claims of the accident year j, plus the change in the estimated reserves for individual claims.

Assumption 1.1 In the following we consider $n+1$ accident years and assume that each claim is either fully settled in the accident year, where the claim occurred, or in one of the following n development years.

Assumption 1.2 We assume throughout that $C_{j,k}$ is non-negative for all $j,k \in \{0,1,\ldots,n\}$, but it may happen that $I_{j,k}$ is negative for some $j \in \{0,1,\ldots,n\}$ and $k \in \{1,\ldots,n\}$ due to estimated reserves for individual claims which turn out to have been too high.

2 Chain–Ladder Method

Consider a portfolio which is described by a family $\{C_{j,k}\}_{j,k\in\{0,1,\ldots,n\}}$ of random variables, where $C_{j,k}$ denotes the claim amount (including estimated reserves for individual claims) of accident year j and development year up to k. We assume that the $C_{j,k}$ are currently observable for $j+k \le n$ and are non-observable for $j+k > n$ with $j,k \in \{0,1,\ldots,n\}$. The method is a simple modification of the chain–ladder method described in [2].

Chain–ladder method:

For every accident year $j \in \{0,1,\ldots,n\}$, the chain–ladder predictor of the expected claim amount is defined by

$$\hat{C}^{\mathrm{CL}}_{j,k} := \begin{cases} C_{j,n-j} \prod_{l=n-j+1}^{k} \hat{f}_l & \text{for } k = n-j+1,\ldots,n, \\ C_{j,n-j} & \text{for } k = n-j, \end{cases} \tag{3}$$

where the so-called age-to-age factor $\hat{f}_k$ is defined by

$$\hat{f}_k := \begin{cases} 1 & \text{if } \sum_{j=0}^{n-k} C_{j,k-1} = 0, \\ \dfrac{\sum_{j=0}^{n-k} C_{j,k}}{\sum_{j=0}^{n-k} C_{j,k-1}} & \text{if } \sum_{j=0}^{n-k} C_{j,k-1} > 0, \end{cases} \tag{4}$$

for every development year $k \in \{1,\ldots,n\}$. The chain–ladder reserve of accident year j is defined by

$$\hat{R}^{\mathrm{CL}}_j := \hat{C}^{\mathrm{CL}}_{j,n} - C_{j,n-j} \tag{5}$$

for $j \in \{0,1,\ldots,n\}$, where adding up the values yields the global chain–ladder reserve $\hat{R}^{\mathrm{CL}}$ for all accident years, i.e.,

$$\hat{R}^{\mathrm{CL}} := \sum_{j=0}^{n} \hat{R}^{\mathrm{CL}}_j. \tag{6}$$

3 Stochastic Chain–Ladder

3.1 Introduction

The history of innovation in claim reserving has usually meant getting better predictors of the expected values of future payment amounts. Less effort had been spent on estimating the distribution of results around the expected value. However, recent discussions of new regulatory and accounting standards have changed the focus to stochastic reserving – the creation of these distributions, remembering, of course, that these distributions are themselves estimates.

Stochastic methods will not and should not replace traditional actuarial methods for claim reserving, but they will provide critical information for making a variety of management decisions.

There has been a large number of papers investigating the statistical basis of the chain–ladder technique, which have made significant advances in the understanding of the chain–ladder method. The aim of this section is to bring these together in a convenient form, and to show how extensions to the models are possible. In this Master thesis we will only consider models for the chain–ladder method – but there are also others, for example linear models or credibility models. For further information to linear and credibility models see [4].

3.2 Multiplicative Model for Increments

The multiplicative model gives a first justification of the chain–ladder method. The model is carried out following [2].

3.2.1 Multiplicative model

There exist parameters $\alpha_0, \alpha_1, \ldots, \alpha_n \in [0, \infty)$ *and* $\vartheta_0, \vartheta_1, \ldots, \vartheta_n \in \mathbb{R}$, *satisfying*

$$\sum_{k=0}^{n} \vartheta_k = 1, \tag{7}$$

such that the increments of the run-off square satisfy

$$\mathbb{E}[I_{j,k}] = \alpha_j \vartheta_k \tag{8}$$

for all $j, k \in \{0, 1, \ldots, n\}$.

Remark 3.1 We get the identities

$$\sum_{k=0}^{n-j} \alpha_j \vartheta_k = \sum_{k=0}^{n-j} \mathbb{E}[I_{j,k}] \quad \textit{for all } j \in \{0, 1, \ldots, n\} \tag{9}$$

and

$$\sum_{j=0}^{n-k} \alpha_j \vartheta_k = \sum_{j=0}^{n-k} \mathbb{E}[I_{j,k}] \quad \textit{for all } k \in \{0, 1, \ldots, n\}. \tag{10}$$

3.2.2 Parameter estimation

To estimate in (8) the unknown parameters $\alpha_0, \alpha_1, \ldots, \alpha_n$ and $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$ we consider random variables $\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_n$ with values in $[0, \infty)$ and real-valued $\hat{\vartheta}_0, \hat{\vartheta}_1, \ldots, \hat{\vartheta}_n$ with

$$\sum_{k=0}^{n} \hat{\vartheta}_k = 1. \tag{11}$$

These random variables are said to be marginal-sum estimators of the parameters $\alpha_0, \alpha_1, \ldots, \alpha_n \in [0, \infty)$ and $\vartheta_0, \vartheta_1, \ldots, \vartheta_n \in \mathbb{R}$ if they solve the marginal equations

$$\sum_{k=0}^{n-j} \hat{\alpha}_j \hat{\vartheta}_k = \sum_{k=0}^{n-j} I_{j,k} \quad \text{for } j \in \{0, 1, \ldots, n\} \tag{12}$$

and

$$\sum_{j=0}^{n-k} \hat{\alpha}_j \hat{\vartheta}_k = \sum_{j=0}^{n-k} I_{j,k} \quad \text{for } k \in \{0, 1, \ldots, n\}. \tag{13}$$

Notice that the equations (12) and (13) have the same structure as the equations (9) and (10), therefore the marginal-sum principle is here a rather natural one. The following theorem gives a satisfying result according to the existence and uniqueness of the marginal-sum estimators.

Theorem 3.2 *Let $\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_n$ be $[0, \infty)$-valued and $\hat{\vartheta}_0, \hat{\vartheta}_1, \ldots, \hat{\vartheta}_n$ be real-valued random variables satisfying (11). We assume that $C_{0,n} > 0$.*

1. *If $\sum_{j=0}^{n-k-1} C_{j,k} > 0$ for $k \in \{1, \ldots, n-1\}$ and if $\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_n$ and $\hat{\vartheta}_0, \hat{\vartheta}_1, \ldots, \hat{\vartheta}_n$ are marginal-sum estimators, then*

$$\hat{\alpha}_j = \hat{C}_{j,n}^{\mathrm{CL}} \tag{14}$$

for all $j \in \{0, 1, \ldots, n\}$ and

$$\hat{\vartheta}_k = \begin{cases} \hat{G}_0 & \text{for } k = 0, \\ \hat{G}_k - \hat{G}_{k-1} & \text{for } k \in \{1, \ldots, n\}, \end{cases} \tag{15}$$

where $\hat{G}_k$ for $k \in \{1, \ldots, n\}$ is defined recursively by

$$\hat{G}_{n-j} := \begin{cases} 1 & \text{for } j = 0, \\ \dfrac{\sum_{l=0}^{j-1} C_{l,n-j}}{\sum_{l=0}^{j-1} \hat{C}_{l,n}^{\mathrm{GU}}} & \text{for } j \in \{1, \ldots, n\} \text{ if } \sum_{l=0}^{j-1} C_{l,n-j} > 0, \\ \hat{G}_{n-j+1} & \text{for } j \in \{1, \ldots, n\} \text{ if } \sum_{l=0}^{j-1} C_{l,n-j} = 0 \end{cases} \tag{16}$$

with

$$\hat{C}_{j,n}^{\mathrm{GU}} := \frac{C_{j,n-j}}{\hat{G}_{n-j}} \tag{17}$$

for all $j \in \{0, 1, \ldots, n\}$.

2. *If (14) and (15) are fulfilled, then $\hat{\alpha}_0, \hat{\alpha}_1, \dots, \hat{\alpha}_n$ and $\hat{\vartheta}_0, \hat{\vartheta}_1, \dots, \hat{\vartheta}_n$ are marginal-sum estimators.*

The marginal-sum estimators exist and are unique.

3.3 Poisson Model

3.3.1 Poisson model

1. *The family $\{I_{j,k}\}_{j,k\in\{0,1,\dots,n\}}$ is independent.*
2. *There exist parameters $\alpha_0, \alpha_1, \dots, \alpha_n \in [0,\infty)$ and $\vartheta_0, \vartheta_1, \dots, \vartheta_n \in \mathbb{R}$, satisfying*
$$\sum_{k=0}^{n} \vartheta_k = 1,$$
such that
$$\mathcal{L}(I_{j,k}) = \mathbf{Poi}(\alpha_j \vartheta_k) \tag{18}$$
holds for all $j,k \in \{0,1,\dots,n\}$.

3.3.2 Parameter estimation

In the Poisson model the joint distribution of all increments is given by

$$\mathbb{P}\Big[\bigcap_{j=0}^{n}\bigcap_{k=0}^{n}\{I_{j,k} = i_{j,k}\}\Big] = \prod_{j=0}^{n}\prod_{k=0}^{n}\left(e^{-\alpha_j\vartheta_k}\frac{(\alpha_j\vartheta_k)^{i_{j,k}}}{i_{j,k}!}\right). \tag{19}$$

To estimate the unknown parameters $\alpha_0, \alpha_1, \dots, \alpha_n$ and $\vartheta_0, \vartheta_1, \dots, \vartheta_n$ we use the maximum-likelihood principle.
Out of (19) we can build the liklihood function

$$L(\hat{\alpha}_0, \hat{\alpha}_1, \dots, \hat{\alpha}_n, \hat{\vartheta}_0, \hat{\vartheta}_1, \dots, \hat{\vartheta}_n) := \prod_{j=0}^{n}\prod_{k=0}^{n-j}\left(e^{-\alpha_j\vartheta_k}\frac{(\alpha_j\vartheta_k)^{I_{j,k}}}{I_{j,k}!}\right).$$

So we get for the log-likelihood function

$$\log L(\hat{\alpha}_0, \hat{\alpha}_1, \dots, \hat{\alpha}_n, \hat{\vartheta}_0, \hat{\vartheta}_1, \dots, \hat{\vartheta}_n) \tag{20}$$
$$= \sum_{j=0}^{n}\sum_{k=0}^{n-j}(\alpha_j\vartheta_k) + I_{j,k}\log(\alpha_j\vartheta_k) - \log(I_{j,k}!)). \tag{21}$$

Taking the derivatives with respect to $\hat{\alpha}_i$ and $\hat{\vartheta}_k$ respectively yield

$$\frac{\partial(\log L)}{\partial\hat{\alpha}_j}(\hat{\alpha}_0, \hat{\alpha}_1, \dots, \hat{\alpha}_n, \hat{\vartheta}_0, \hat{\vartheta}_1, \dots, \hat{\vartheta}_n) = \sum_{k=0}^{n-j}\left(-\hat{\vartheta}_k + I_{j,k}\frac{1}{\hat{\alpha}_j}\right)$$

$$\frac{\partial(\log L)}{\partial\hat{\vartheta}_k}(\hat{\alpha}_0, \hat{\alpha}_1, \dots, \hat{\alpha}_n, \hat{\vartheta}_0, \hat{\vartheta}_1, \dots, \hat{\vartheta}_n) = \sum_{j=0}^{n-k}\left(-\hat{\alpha}_j + I_{j,k}\frac{1}{\hat{\vartheta}_k}\right).$$

These yield maximum-likelihood estimates of $\hat{\alpha}_j, \hat{\vartheta}_k$ with $j,k \in \{0,1,\ldots,n\}$ satisfy the following relations

$$\hat{\alpha}_j = \sum_{k=0}^{n-j} \frac{I_{j,k}}{\hat{\vartheta}_k} \iff \sum_{k=0}^{n-j} \hat{\alpha}_j \hat{\vartheta}_k = \sum_{k=0}^{n-j} I_{j,k} \tag{22}$$

$$\hat{\vartheta}_k = \sum_{j=0}^{n-k} \frac{I_{j,k}}{\hat{\alpha}_j} \iff \sum_{j=0}^{n-k} \hat{\alpha}_j \hat{\vartheta}_k = \sum_{j=0}^{n-k} I_{j,k}. \tag{23}$$

Remark 3.3 Due to (22) and (23) the estimators $\hat{\alpha}_j, \hat{\vartheta}_k$ of α_j, ϑ_k fulfill the marginal-sum equations (12) and (13). Therefore Theorem 3.2 also holds for the Poisson model.

3.4 Non-Parametric Chain–Ladder

In the multinomial model a special distribution of the $I_{j,k}$ is assumed, but the $I_{j,k}$ can also be modeled in a non-parametric form, which yields the estimation $\hat{f}_k$ defined by (4) for $k \in \{1,\ldots,n\}$ and $\hat{C}_{j,k}$ defined by (3) for $j,k \in \{0,1,\ldots,n\}$ with $j+k \geq n+1$, irrespective of the distribution of the $I_{j,k}$. The model will be carried out following [1].

Throughout this section all equalities involving conditional expectations are understood to hold almost surely with respect to the probability measure $\mathbb{P}$. We will carry out this study under the following assumptions.

Assumption 3.4 There exist parameters $f_1,\ldots,f_n \in [0,\infty)$, such that

$$\mathbb{E}[C_{j,k}|C_{j,0},\ldots,C_{j,k-1}] = C_{j,k-1} f_k$$

holds for all $j \in \{0,1,\ldots,n\}$ and $k \in \{1,\ldots,n\}$.

Assumption 3.5 The variables $C_{j,k}$ of different accident years, i.e.

$$\{C_{j,0},\ldots,C_{j,n}\}, \{C_{l,0},\ldots,C_{l,n}\} \text{ for } j \neq l \text{ are independent.}$$

Assumption 3.6 The random variables are square-integrable and there exist unknown parameters $\sigma_1^2,\ldots,\sigma_n^2 \in [0,\infty)$ such that

$$\mathbb{V}\text{ar}[C_{j,k}|C_{j,0},\ldots,C_{j,k-1}] = C_{j,k-1}\sigma_k^2,$$

for all $j \in \{0,1,\ldots,n\}$, $k \in \{1,\ldots,n\}$.

Assumption 3.7

$$\sum_{j=0}^{n-k-1} C_{j,k} > 0 \qquad \text{for every development year } k \in \{0,1,\ldots,n-1\}.$$

Assumption 3.8 For every $j \in \{0,1,\ldots,n\}$ and $k \in \{0,1,\ldots,n-1\}$ with $j+k \leq n$ we assume $C_{j,k} > 0$.

The following theorem shows, that 3.4 and 3.5 are indeed the suitable assumptions for the chain–ladder method.

Theorem 3.9 *Let $\Delta = \{C_{j,k} \,|\, j+k \leq n\}$ be the set of all data observed so far. Under the assumptions 3.4 and 3.5*

$$\begin{aligned}\mathbb{E}[C_{j,n}|\Delta] &= \mathbb{E}[C_{j,n}|C_{j,0},\ldots,C_{j,n-j}] \\ &= C_{j,n-j} \cdot f_{n-j+1}\cdots f_n\end{aligned}$$

holds for all $j \in \{1,\ldots,n\}$.

The next theorem shows another consequence of 3.4 and 3.5, namely the uncorrelatedness and unbiasedness of the estimators $\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_n$ of $f_1, f_2, \ldots, f_n$.

Theorem 3.10 *Under the assumptions 3.4, 3.5 and 3.7 the age-to-age factors $\hat{f}_1, \ldots, \hat{f}_n$, defined by (4), are unbiased and uncorrelated.*

Calculation of the mean squared error

The aim of the chain–ladder method and every other claim reserving method is to give a forecast of the ultimate claim amount $C_{j,n}$ for the accident years $j = 1,\ldots,n$. All these methods only yield a point estimate for $C_{j,n}$ which will normally turn out to be more or less wrong. Therefore it will be of great interest to determine the mean squared error $\mathrm{mse}\big[\hat{C}_{j,n}\big]$ of the predictor $\hat{C}_{j,n}$ of $C_{j,n}$, which is defined by

$$\mathrm{mse}\big[\hat{C}_{j,n}\big] = \mathbb{E}\big[\big(\hat{C}_{j,n} - C_{j,n}\big)^2\big|\Delta\big], \tag{24}$$

where $\Delta = \{C_{j,k} \,|\, j+k \leq n\}$ is the set of all data observed so far.

Theorem 3.11 *Under the assumptions 3.4, 3.5, 3.6 and 3.7*

1. *the estimator $\hat{\sigma}_k^2$ of σ_k^2, given by*

$$\hat{\sigma}_k^2 = \frac{1}{n-k}\sum_{j=0}^{n-k} C_{j,k-1}\left(\frac{C_{j,k}}{C_{j,k-1}} - \hat{f}_k\right)^2 \tag{25}$$

   for $k \in \{1,\ldots,n-1\}$, is unbiased and

2. *the mean squared error $\mathrm{mse}\big[\hat{R}_j\big]$ for $j \in \{1,\ldots,n\}$ can be estimated by*

$$\widehat{\mathrm{mse}}\big[\hat{R}_j\big] = \hat{C}_{j,n}^2 \sum_{k=n-j+1}^{n} \frac{\hat{\sigma}_k^2}{\hat{f}_k^2}\left(\frac{1}{\hat{C}_{j,k-1}} + \frac{1}{\sum_{h=0}^{n-k} C_{h,k-1}}\right),$$

   where $\hat{C}_{j,k-1} = C_{j,n-j}\hat{f}_{n-j+1}\cdots\hat{f}_{k-1}$ for $k > n-j$ are the estimated values of the future $C_{j,k-1}$ and $\hat{C}_{j,n-j} = C_{j,n-j}$.

Remark 3.12 In (25) the estimator $\hat{\sigma}_k^2$ of σ_k^2 is only given for $k \in \{1,\ldots,n-1\}$.

- If $\hat{f}_n = 1$ and if the claims development is assumed to be finished after n years, $\hat{\sigma}_n^2$ can be assumed to be zero.
- Otherwise an estimator of σ_n^2 can be obtained by extrapolating the series $\hat{\sigma}_1, \ldots, \hat{\sigma}_{n-2}, \hat{\sigma}_{n-1}$ by one further member. In [1] Mack proposes one possibility to do this, namely he requires that

$$\frac{\hat{\sigma}_{n-2}}{\hat{\sigma}_{n-1}} = \frac{\hat{\sigma}_{n-1}}{\hat{\sigma}_n}$$

holds at least for $\hat{\sigma}_{n-2} > \hat{\sigma}_{n-1}$. Therefore $\hat{\sigma}_n$ can be estimated by

$$\hat{\sigma}_n^2 = \min\left(\frac{\hat{\sigma}_{n-1}^4}{\hat{\sigma}_{n-2}^2}; \min(\hat{\sigma}_{n-2}^2, \hat{\sigma}_{n-1}^2)\right).$$

Theorem 3.13 *Under assumptions 3.4, 3.5 and 3.6 the mean squared error of the predictor*

$$\hat{R} = \hat{R}_1 + \cdots + \hat{R}_n$$

for the global reserve

$$R = R_1 + \cdots + R_n$$

can be estimated by

$$\widehat{\mathrm{mse}}\big[\hat{R}\big] = \sum_{j=1}^{n}\left(\mathrm{mse}\big[\hat{R}_j\big] + \hat{C}_{j,n}\left(\sum_{l=j+1}^{n} \hat{C}_{l,n}\right) \sum_{k=n-j+1}^{n} \frac{2\hat{\sigma}_k^2/\hat{f}_k^2}{\sum_{l=1}^{n-k} C_{l,k-1}}\right). \qquad (26)$$

4 Multivariate Methods

4.1 Notation

Let $m \in \mathbb{N}$ be the number of subportfolios and for $p \in \{1, \ldots, m\}$

$$C_{j,k}^{(p)}$$

denotes the ultimate claim size of accident year $j \in \{0, 1, \ldots, n\}$ and development year $k \in \{0, 1, \ldots, n\}$. Furthermore, let

$$f_{j,k}^{(p)} = \frac{C_{j,k}^{(p)}}{C_{j,k-1}^{(p)}}$$

denote the individual development factor of accident year $j \in \{0, 1, \ldots, n\}$ and development year $k \in \{1, \ldots, n\}$. We will carry out this model under the assumption that all $C_{j,k}^{(p)} > 0$ for $j, k \in \{0, 1, \ldots, n\}$ and $p \in \{1, \ldots, m\}$.

Analogous to the case of one run-off square the $C_{j,k}^{(p)}$ are observable for $j+k \leq n$ and non-observable for $j + k > n$. For $j, k \in \{0, 1, \ldots, n\}$ we thus obtain the m-dimensional random vector of ultimate claims

$$C_{j,k} = \begin{pmatrix} C_{j,k}^{(1)} \\ \vdots \\ C_{j,k}^{(m)} \end{pmatrix}$$

and

$$f_{j,k} = \begin{pmatrix} f_{j,k}^{(1)} \\ \vdots \\ f_{j,k}^{(m)} \end{pmatrix}$$

is the m-dimensional random vector of individual development factors. We consider a model involving successive conditioning with respect to the σ-algebras $\mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_{n-1}$, where for each development year $k \in \{1, \ldots, n\}$ the σ-algebra

$$\mathcal{G}_{k-1}$$

represents the information provided by the claim amount $C_{j,l}$, with $j \in \{0, 1, \ldots, n-k+1\}$ and $l \in \{0, 1, \ldots, k-1\}$.
In the following it is better to represent the random vector $C_{j,k}$ by the diagonal matrix

$$T_{j,k} := \text{diag}(C_{j,k}) = \begin{pmatrix} C_{j,k}^{(1)} & & 0 \\ & \ddots & \\ 0 & & C_{j,k}^{(m)} \end{pmatrix}.$$

Obviously,

$$C_{j,k} = T_{j,k} \cdot I$$

holds for all $j, k \in \{0, 1, \ldots, n\}$, where I is the vector in $\mathbb{R}^m$ with all coordinates being equal to 1. Furthermore

$$C_{j,k} = T_{j,k-1}\ f_{j,k} \tag{27}$$

holds for all $j \in \{0, 1, \ldots, n\}$ and $k \in \{1, \ldots, n\}$.

4.2 Assumptions

For each development year $k \in \{1, \ldots, n\}$, there exist a $\mathcal{G}_{k-1}$-measurable, m-dimensional random vector f_k and a random matrix V_k, which is symmetric and positive definite, such that

$$\mathbb{E}[C_{j,k}|\mathcal{G}_{k-1}] = T_{j,k-1} f_k \tag{28}$$

and

$$\mathbb{V}\text{ar}[C_{j,k}|\mathcal{G}_{k-1}] = T_{j,k-1}^{\frac{1}{2}} V_k T_{j,k-1}^{\frac{1}{2}} \tag{29}$$

holds for $j \in \{0, 1, \ldots, n\}$, $k \in \{1, \ldots, n\}$ and

$$\text{Cov}(C_{j,k}, C_{l,k}|\mathcal{G}_{k-1}) = 0^1 \tag{30}$$

holds for $j, l \in \{0, 1, \ldots, n\}$ with $l \neq j$.
In the following we assume that the assumptions of Subsection 4.2 are fulfilled.

[1] 0 denotes the matrix in $\mathbb{R}^m$ with all coordinates being equal to 0.

4.3 Multivariate chain–ladder method

The chain–ladder method proposed in Section 2 is defined as an algorithm without an underlying stochastic model whereas the multivariate chain–ladder method presented now is based on a stochastic model. The following multivariate chain–ladder method consists of n conditional linear models corresponding to the development years $k \in \{1, \dots, n\}$. To understand the multivariate chain–ladder method in [3] Schmidt gives a good justification. For a fixed development year $k \in \{1, \dots, n\}$, let C_1 denote a block vector consisting of the random vector $C_{j,k}$, with $j \leq n-k$ and let A_1 denote a block matrix consisting of the random matrices $T_{j,k}$, with $j \leq n+k$. Furthermore let

$$C_2 := C_{n-k+1,k}$$

and

$$A_2 := T_{n-k+1,k}.$$

Then the random vectors C_1 and C_2 and the random matrices A_1 and A_2 depend on the development year k and we have

$$\begin{aligned}\mathbb{E}[C_1|\mathcal{G}_{k-1}] &= A_1 f_k \\ \mathbb{E}[C_2|\mathcal{G}_{k-1}] &= A_2 f_k.\end{aligned}$$

Thus, the multivariate chain–ladder model consist indeed of n conditional linear models.

Multivariate chain–ladder method:

Under the assumptions of Subsection 4.2 the multivariate chain–ladder method is defined by

$$\hat{C}^{\mathrm{CL}}_{j,n-j} := C_{j,n-j}$$

for all $j \in \{1, \dots, n\}$ *and*

$$\hat{f}^{\mathrm{CL}}_k := \left(\sum_{j=0}^{n-k} T^{\frac{1}{2}}_{j,k-1} V_k^{-1} T^{\frac{1}{2}}_{j,k-1} \right)^{-1} \sum_{j=0}^{n-k} (T^{\frac{1}{2}}_{j,k-1} V_k^{-1} T^{\frac{1}{2}}_{j,k-1}) T^{-1}_{j,k-1} C_{j,k}$$

for all $k \in \{1, \dots, n\}$ *as well as*

$$\begin{aligned}C^{\mathrm{CL}}_{j,k} &:= T^{\mathrm{CL}}_{j,k-1} \hat{f}^{\mathrm{CL}}_k \\ T^{\mathrm{CL}}_{j,k-1} &:= \mathrm{diag}\left(C^{\mathrm{CL}}_{j,k-1}\right)\end{aligned}$$

for all $j, k \in \{1, \dots, n\}$*, with* $j + k > n$.

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